Hydromagnetics of a Spherical Conductor

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IN Refs. 1 and 2 the hydromagnetics of a perfectly conducting sphere moving slowly through a magnetic field in an arbitrary direction was reconsidered. Stewartson's result[3] is that the liquid inside the cylinder C, circumscribing the sphere with generators parallel to the applied magnetic field (z), moves with the sphere as if solid, while the flow outside C is cylindrical but complicated. Correction was made of his treatment and, in particular, of this last flow pattern.

However, as it stands the result is of limited interest, and it is the object of this note to show that a sphere of arbitrary nonzero conductivity ultimately behaves in the same way (in contrast to a nonconducting sphere[1,2]). In addition, the leading term in the decay of the magnetic disturbance, which is not the same as for a perfect conductor, will be obtained.

As in Ref. 1,‡ the steady-state equations

$$\left.\begin{aligned} \partial\mathbf{h}/\partial z &= \operatorname{grad}\phi \\ \operatorname{div}\mathbf{v} &= 0 \\ \partial\mathbf{v}/\partial z &= \beta \operatorname{curl}\operatorname{curl}\mathbf{h} \\ \operatorname{div}\mathbf{h} &= 0 \\ \mathbf{E} &= \mathbf{k}\times\mathbf{v} + \beta\operatorname{curl}\mathbf{h} \end{aligned}\right\} \quad (1)$$

do not determine the ultimate flow pattern. However, they do imply that the magnetic disturbance $\mathbf{h}$ is ultimately zero. For, as in Ref. 1,

$$\operatorname{curl}\operatorname{curl}\mathbf{h} = 0 \quad (2)$$

in the fluid so that, if it is assumed that the current curl$\mathbf{h}$ vanishes at infinity in planes z = const, a simple argument using the first of Eqs. (1) shows that

$$\operatorname{curl}\mathbf{h} = 0 \quad (3)$$

everywhere in the fluid. However, (2) also holds inside the sphere, whatever its conductivity, so that curl$\mathbf{h}$ = grad ϕ where $\nabla^2\phi = 0$. Since curl$_n\mathbf{h}$ is continuous across the sphere, ϕ = const and

$$\operatorname{curl}\mathbf{h} = 0 \qquad \operatorname{div}\mathbf{h} = 0 \quad (4)$$

inside as well as outside the sphere. Together with the continuity of the magnetic field across the surface this implies that $\mathbf{h} = 0$ everywhere.

In order to determine the ultimate flow pattern when the sphere ($R \equiv [x^2 + y^2 + z^2]^{1/2} = 1$) moves at right angles (x) to the magnetic field, it is sufficient to show that ultimately $v_z = 0$ on the sphere. Then the boundary conditions on the governing Eqs. (7–9) of Ref. 1 are complete and indeed the same as in the perfectly conducting case. For this purpose, note that the fifth of Eqs. (1) and Eq. (3) give the tangential electric field $E_t = (xv_x + yv_y)/r$ in the fluid, while from Eq. (4) $E_t = x/r$ in the sphere,§ since it is a conductor. Here $r = (x^2 + y^2)^{1/2}$. Continuity of E_t across the surface of the sphere therefore yields $xv_x + yv_y = x$ while continuity of the normal velocity gives $xv_x + yv_y + zv_z = x$, so that $v_z = 0$ on the surface.

Now consider how $\mathbf{h} \to 0$. The two-dimensional parts $\mathbf{h}^t$ are determined with $\mathbf{v}$ and hence are the same as for a perfect conductor (see Appendix to Ref. 1). The discontinuity in $\mathbf{h}^t$ for $r > 1$ on $z = 0$ is the same as that for the pair of two-dimensional potential fields $\pm\, 4/(3\pi^{3/2}\beta^{1/2}t^{1/2})$ grad (x/r^2) ($z \gtrless 0$). It may be removed by adding the pair of three-dimensional potential fields $4/(3\pi^{3/2}\beta^{1/2}t^{1/2})$ grad $(xz/r^2R \mp x/r^2)$, i.e.,

$$\left.\begin{aligned} h_x{}^p &= -4/(3\pi^{3/2}\beta^{1/2}t^{1/2})[(x^2 - y^2)(z/R \mp 1)/r^4 + x^2z/r^2R^3] \\ h_y{}^p &= -4/(3\pi^{3/2}\beta^{1/2}t^{1/2})[2xy(z/R \mp 1)/r^4 + xyz/r^2R^3] \\ h_z{}^p &= 4/(3\pi^{3/2}\beta^{1/2}t^{1/2})x/R^3 \end{aligned}\right\} \quad (5)$$

everywhere outside the sphere. This is, in fact, the correct additional field since the total field $\mathbf{h}^t + \mathbf{h}^p$ passes continuously into the field

$$\left.\begin{aligned} h_x &= -8z/(3\pi^{3/2}\beta^{1/2}t^{1/2}) \\ h_y &= 0 \\ h_z &= 4x/(3\pi^{3/2}\beta^{1/2}t^{1/2}) \end{aligned}\right\} \quad (6)$$

satisfying (2) inside the sphere.

Note that (5) and (6), and hence the uniform current $(0, -4/\pi^{3/2}\beta^{1/2}t^{1/2}, 0)$ flowing through the sphere, are independent of the conductivity of the sphere (β is essentially the magnetic diffusivity of the liquid). On the other hand, the corresponding formulas for both a nonconducting and a perfectly conducting sphere are quite different (see Ref. 1).

The derivation in Ref. 2 of the ultimate flow pattern for motion along the magnetic field did not involve the conductivity of the sphere. Hence, the same $\mathbf{v}$ and $\mathbf{h}^t$ obtain in the present case: these variables are independent of the conductivity, be it zero, finite, or infinite. The problem of determining the potential parts $\mathbf{h}^p$ outside and the potential field‖ $\mathbf{h}$ inside is the same as for an insulator: the total magnetic field must be continuous across the surface. The formulas are given in Sec. 4 of Ref. 2.

In a forthcoming paper similar and other results will be developed for an ellipsoid.

References

[1] Ludford, G. S. S. and Singh, M. P., "The motion of a nonconducting sphere through a conducting fluid in a magnetic cross-field," Proc. Cambridge Phil. Soc. 59, 615–624 (1963).

[2] Ludford, G. S. S. and Singh, M. P., "On the motion of a sphere through a conducting fluid in the presence of a magnetic field," Proc. Cambridge Phil. Soc. 59, 625–635 (1963).

[3] Stewartson, K., "Motion of a sphere through a conducting fluid in the presence of a strong magnetic field," Proc. Cambridge Phil. Soc. 52, 301–316 (1956).

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‡ Notation is the same as in Refs. 1 and 2 and familiarity with these papers is assumed. All variables are nondimensional in the obvious way. Axes are fixed with respect to the undisturbed fluid, but with origin instantaneously coincident with the center of the sphere.

§ This is the crux of the matter: ultimately there are no currents in the sphere so that it behaves like a perfect conductor.

‖ Because of the axial symmetry, the currents form circles which do not penetrate the sphere.